International Journal of Scientific & Engineering Research Volume 8, Issue 8, August-2017
ISSN 2229-5518

A Note on Involution Pseudoknot-bordered Words

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Abstract

This paper continues the exploration of properties concerning involution pseudoknot-(un)bordered words for a morphic involution or an antimorphic involution. Involution pseudoknot-(un)bordered words are a generalization of the classical notions of bordered and unbordered words. There are some results obtained in this paper. Let θ be an antimorphic involution. We prove that, under some conditions, $L_{\mathrm{d}}^{\theta}(w) = L_{\mathrm{cd}}^{\theta}(w)$ where $L_{\mathrm{d}}^{\theta}(w)$ denotes the set of all proper θ-borders of a nonempty word w and $L_{\mathrm{cd}}^{\theta}(w)$ denotes the set of all θ-pseudoknot-borders of a nonempty word w. We also get that for a nonempty skew θ-palindrome word, it is θ-pseudoknot-bordered, and find that for every θ-pseudoknot-unbordered word w, the root of w is also θ-pseudoknot-unbordered. Moreover, we show that $D_\theta(i)$ and $K_\theta(i)$ for every $i \geq 2$ are dense, where $D_\theta(i)$ denotes the sets of words that have exactly i θ-borders and $K_\theta(i)$ denotes the sets of words that have exactly i θ-pseudoknot-borders.

Keywords: Palindrome, Skew Involution Palindrome, Involution Unbordered Words, Dense.

1 Introduction

We view a DNA single strand as a string over the DNA alphabet of bases $\{A, C, G, T\}$. DNA single strands have a necessary biochemical property which is the Watson-Crick complementarity, wherein A can bind to T and C can bind to G. DNA single strands which are the Watson-Crick complementarity often result the secondary structures. The secondary structures of DNA single strands are either folding onto themselves to form intra-molecular structure, or interacting with each other to form inter-molecular structure. When encoding data on DNA strands, it needs to avoid the secondary structures of DNA strands because the secondary structures make DNA strands unavailable for biocomputations. Consequently, there are many studies [8-11] relating to the property of unwanted intra-molecular and inter-molecular structures from many different points of view. Kari and Mahalingam ([4]) introduced and investigated the concept of a θ-unbordered word that avoid some of common inter-molecular and intra-molecular structures where θ is an antimorphic involution. A θ-bordered word w is a nonempty word which has a proper prefix u, and a proper suffix $\theta(u)$, that is, $w = u\alpha = \beta\theta(u)$ for some nonempty words α, β. The notions of θ-bordered and θ-unbordered words are generalizations of classical notions in combinatorics of words. Furthermore, the pseudoknot is formed by RNA strands, it is another significant intra-molecular structure. The type of pseudoknot found in E. Coli transfer-messenger-RNA ([3]). It can be modeled as a nonempty word w using the form $u_1xu_2yu_3\theta(x)u_4\theta(y)u_5$, that

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is, $w = u_1xu_2yu_3\theta(x)u_4\theta(y)u_5$. In the design of DNA strands used for computational purposes, the involution pseudoknot-unbordered words are models of DNA or RNA strands that will not form pseudoknot inter-molecular and intra-molecular structures. We will investigate involution pseudoknot-bordered (or θ-pseudoknot-bordered) words and involution pseudoknot-unbordered (or θ-pseudoknot-bordered) words for an antimorphic involution θ in this paper. A nonempty word w is θ-pseudoknot-bordered if $w = xy\alpha = \beta\theta(yx)$ for some nonempty words x, y, α, β ([8]). Note that this is a special case of the general model of pseudoknots with u_1, u_2, u_4, u_5 being empty words.

This paper consists of four sections. The first section is an overview of this study. The second section includes some well-known definitions and applied properties. We observe some examples to get the concept concerning involution pseudoknot-(un)bordered words and involution (un)bordered words in the third section. Form those observations, we prove that, under some conditions, $L^{\theta}_{\mathrm{d}}(w) = L^{\theta}_{\mathrm{cd}}(w)$. We find that every nonempty skew θ-palindrome word is θ-pk-bordered. Since every θ-palindrome word is a skew θ-palindrome word, this follows that θ-palindrome word is also θ-pk-bordered. We get that for every θ-pk-unbordered word w, the root of w is also a θ-pk-unbordered when θ is an antimorphic involution on X^*. A procedure to create θ-pk-bordered words is provided. We study the density of involution bordered languages and involution pseudoknot-bordered languages in the final section. It can be proved that $D_\theta(i)$ and $K_\theta(i)$ for every $i \geq 1$ are dense.

2 Preliminaries

Let X be a finite alphabet and X^* be the free monoid generated by X. Any element of X^* is called a *word*. The length of a word w is denoted by $\lg(w)$. Any subset of X^* is called a *language*. Let $X^+ = X^* \setminus \{\lambda\}$, where λ is the empty word. Let $|L|$ denote the cardinality of language L. A *primitive word* is a word which is not a power of any other word. Let Q be the set of all primitive words over X. Every word $u \in X^+$ can be expressed as a power of a primitive word in a unique way, that is, for any $u \in X^+$, $u = f^n$ for a unique $f \in Q$ and $n \geq 1$. In this case, f is the primitive root of u and denoted by $\sqrt{u} = f$. Let $u = a_1a_2\cdots a_n$ where $a_i \in X$. The reverse of the word u is $u^R = a_n \cdots a_2a_1$. A word u is called *palindrome* if $u = u^R$. Let R be the set of all palindrome words over X. A word $u \in X^*$ is said to be a *border* of a word $w \in X^*$ if $w = xu = uy$ for some $x, y \in X^*$. A word $u \in X^*$ is a *conjugate* of another word $w \in X^*$ if there exists $v \in X^*$ such that $uv = vw$. The partial order relation $\leq_{\mathrm{p}}$ (resp., $\leq_{\mathrm{s}}$) is defined as: for $u, v \in X^*$, $v \leq_{\mathrm{p}} u$ (resp., $v \leq_{\mathrm{s}} u$) if and only if $u \in vX^*$ (resp., $u \in X^*v$). Moreover, the partial order relation $<_{\mathrm{p}}$ (resp., $<_{\mathrm{s}}$) is defined as: for $u, v \in X^+$, $v <_{\mathrm{p}} u$ (resp., $v <_{\mathrm{s}} u$) if and only if $u \in vX^+$ (resp., $u \in X^+v$).

An *involution* $\theta : S \to S$ of $S \subseteq X^*$ is a mapping such that $\theta^2 = I$ where I is the identity mapping. A mapping $\alpha : X^* \to X^*$ is a morphism of X^* if for all $u, v \in X^*$, $\alpha(uv) = \alpha(u)\alpha(v)$ or an antimorphism of X^* if $\alpha(uv) = \alpha(v)\alpha(u)$. For instance, let $X = \{A, C, G, T\}$ and $u = AACGT$. If θ is an antimorphic involution on X^* which maps A to T, C to G, and vice versa, then $\theta(u) = \theta(T)\theta(G)\theta(C)\theta(A)\theta(A) = ACGTT$. If θ is a morphic involution on X^* which maps A to T, C to G, and vice versa, then $\theta(u) = \theta(A)\theta(A)\theta(C)\theta(G)\theta(T) = TTGCA$. We now recall some definitions introduced by Kari and Mahalingam ([4]-[7]). Throughout the paper we assume that the alphabet X is such that $|X| \geq 2$ and the involution θ is not the identity function.

Definition 2.1 *Let θ be either a morphic or an antimorphic involution on X^*. A word $w \in X^*$ is a θ-conjugate of another word $u \in X^*$ if $uv = \theta(v)w$ for some $v \in X^*$.*

For instance, let $X = \{A, C, G, T\}$, $u = ACT$, and θ is an antimorphic involution on X^* which maps A to T, C to G, and vice versa. Then the set of θ-conjugates of u is $\{ACT, CTT, TGT, AGT\}$.

Definition 2.2 *Let θ be an antimorphic involution on X^*.*

(1) A word $w \in X^*$ is called a θ-palindrome word if $w = \theta(w)$.

(2) A word w is said to be skew θ-palindrome if $w = xy$ implies that $\theta(w) = yx$ for some $x, y \in X^*$.

Note that every θ-palindrome word is a skew θ-palindrome word because any θ-palindrome word as a product of itself and the empty word λ. The inverse statement is not true. For instance, let $X = \{A, C, G, T\}$ and θ is an antimorphic involution on X^* which maps A to T, C to G, and vice versa. A word $ATAT$ is θ-palindrome and is also a skew θ-palindrome word. Let $w = ATCG$. Then w is a skew θ-palindrome word, but it is not θ-palindrome.

3 Involution pseudoknot-bordered words

In this section we study some properties of involution pseudoknot-bordered (or θ-pseudoknot-bordered) words for a morphic involution or an antimorphic involution θ on X^*. The notion of θ-pseudoknot-bordered word ([5],[8]) is a proper generalization of the notion of θ-bordered word ([4],[6]). Let θ be a morphic involution or an antimorphic involution on X^*. A word $u \in X^*$ is said to be a proper θ-border of a word $w \in X^+$ if u is a proper prefix of w and $\theta(u)$ is a proper suffix of w, i.e., $w = u\alpha = \beta\theta(u)$ for some $\alpha, \beta \in X^+$. Let $L^\theta_{\mathrm{d}}(w)$ denote the set of all proper θ-borders of a word $w \in X^+$. Note that $\lambda \in L^\theta_{\mathrm{d}}(w)$ for all $w \in X^+$. A word $w \in X^+$ is said to be θ-bordered if it has a proper θ-border other than λ, i.e., $|L^\theta_{\mathrm{d}}(w)| \geq 2$; otherwise, it is θ-unbordered. Moreover, $D_\theta(i) = \{w \in X^+ | \ |L^\theta_{\mathrm{d}}(w)| = i\}$ for every $i \geq 1$ where $D_\theta(1)$ is the set of all θ-unbordered words.

For a word $u \in X^*$, a word $v \in X^*$ is called a *cyclic permutation* of u if there exist two words $x, y \in X^*$ such that $u = xy$ and $v = yx$. A word $u \in X^*$ is called a θ-pseudoknot-border (or θ-pk-border) of a word $w \in X^+$ if there exists a cyclic permutation v of u such that $w = u\alpha' = \beta'\theta(v)$ for some $\alpha', \beta' \in X^*$. Then we have $w = xy\alpha' = \beta'\theta(yx)$ for some $\alpha', \beta' \in X^*$. Let $L^\theta_{\mathrm{cd}}(w)$ denote the set of all θ-pk-borders of a word $w \in X^+$ and $K_\theta(i) = \{w \in X^+ | \ |L^\theta_{\mathrm{cd}}(w)| = i\}$ for every $i \geq 1$. A nonempty word is θ-pseudoknot-bordered (or θ-pk-bordered) if it has a nonempty θ-pk-border; otherwise, it is θ-pseudoknot-unbordered. Note that $\lambda \in L^\theta_{\mathrm{cd}}(w)$ for all $w \in X^+$. Then $K_\theta(1)$ is the set of all θ-pk-unbordered words.

Example 1 Let $X = \{a, b\}$ be an alphabet. Let θ be an antimorphic involution on X^* with $\theta(a) = b$ and $\theta(b) = a$.

(1) Let $w_1 = aabb$. Note that w_1 is a θ-palindrome word. We study the following procedure to get $L^\theta_{\mathrm{d}}(w_1)$.

- $w_1 = \lambda \cdot aabb = aabb \cdot \theta(\lambda)$; $w_1 = a \cdot abb = aab \cdot \theta(a)$; $w_1 = aa \cdot bb = aa \cdot \theta(aa)$; $w_1 = aab \cdot b = a \cdot \theta(aab)$.

Then $L^\theta_{\mathrm{d}}(w_1) = \{\lambda, a, aa, aab\}$.

Moreover, we study the following procedure to get $L^\theta_{\mathrm{cd}}(w_1)$.

- $w_1 = \lambda \cdot aabb = aabb \cdot \theta(\lambda)$.

- $w_1 = \underline{\lambda} \cdot \underline{a} \cdot abb = aab \cdot \underline{\theta(\lambda)} \cdot \underline{\theta(a)}$; $\quad w_1 = \underline{a} \cdot \underline{\lambda} \cdot abb = aab \cdot \underline{\theta(a)} \cdot \underline{\theta(\lambda)}$.
- $w_1 = \underline{\lambda} \cdot \underline{aa} \cdot bb = aa \cdot \underline{\theta(\lambda)} \cdot \underline{\theta(aa)}$; $\quad w_1 = \underline{a} \cdot \underline{a} \cdot bb = aa \cdot \underline{\theta(a)} \cdot \underline{\theta(a)}$;
  $w_1 = \underline{aa} \cdot \underline{\lambda} \cdot bb = aa \cdot \underline{\theta(aa)} \cdot \underline{\theta(\lambda)}$.
- $w_1 = \underline{\lambda} \cdot \underline{aab} \cdot b = a \cdot \underline{\theta(\lambda)} \cdot \underline{\theta(aab)}$; $\quad w_1 = \underline{a} \cdot \underline{ab} \cdot b \neq a \cdot \underline{\theta(a)} \cdot \underline{\theta(ab)} = abab$;
  $w_1 = \underline{aa} \cdot \underline{b} \cdot b \neq a \cdot \underline{\theta(aa)} \cdot \underline{\theta(b)} = abba$; $\quad w_1 = \underline{aab} \cdot \underline{\lambda} \cdot b = a \cdot \underline{\theta(aab)} \cdot \underline{\theta(\lambda)}$.
- $w_1 = \underline{\lambda} \cdot \underline{aabb} \cdot \lambda = \lambda \cdot \underline{\theta(\lambda)} \cdot \underline{\theta(aabb)}$; $\quad w_1 = \underline{a} \cdot \underline{abb} \cdot \lambda \neq \lambda \cdot \underline{\theta(a)} \cdot \underline{\theta(abb)} = baab$;
  $w_1 = \underline{aa} \cdot \underline{bb} \cdot \lambda \neq \lambda \cdot \underline{\theta(aa)} \cdot \underline{\theta(bb)} = bbaa$; $\quad w_1 = \underline{aab} \cdot \underline{b} \cdot \lambda \neq \lambda \cdot \underline{\theta(aab)} \cdot \underline{\theta(b)} = abba$;
  $w_1 = \underline{aabb} \cdot \underline{\lambda} \cdot \lambda = \lambda \cdot \underline{\theta(aabb)} \cdot \underline{\theta(\lambda)}$.

Then $L^{\theta}_{\mathrm{cd}}(w_1) = \{\lambda, a, aa, aab, aabb\}$. Thus $L^{\theta}_{\mathrm{d}}(w_1) \subset L^{\theta}_{\mathrm{cd}}(w_1)$. We have $w_1 \in D_\theta(4)$ and $w_1 \in K_\theta(5)$.

(2) Let $w_2 = aab$. $L^{\theta}_{\mathrm{d}}(w_2) = \{\lambda, a\}$ and $L^{\theta}_{\mathrm{cd}}(w_2) = \{\lambda, a\}$. Thus $L^{\theta}_{\mathrm{d}}(w_2) = L^{\theta}_{\mathrm{cd}}(w_2)$. We have $w_2 \in D_\theta(2)$ and $w_2 \in K_\theta(2)$.

(3) Let $w_3 = aa$. $L^{\theta}_{\mathrm{d}}(w_3) = \{\lambda\}$ and $L^{\theta}_{\mathrm{cd}}(w_3) = \{\lambda\}$. Thus $L^{\theta}_{\mathrm{d}}(w_3) = L^{\theta}_{\mathrm{cd}}(w_3)$. We have $w_3 \in D_\theta(1)$ and $w_3 \in K_\theta(1)$.

(4) Let $w_4 = aba$. $L^{\theta}_{\mathrm{d}}(w_4) = \{\lambda\}$ and $L^{\theta}_{\mathrm{cd}}(w_4) = \{\lambda, ab\}$. Thus $L^{\theta}_{\mathrm{d}}(w_4) \subset L^{\theta}_{\mathrm{cd}}(w_4)$. We have $w_4 \in D_\theta(1)$ and $w_4 \in K_\theta(2)$, that is, $w_4 \notin K_\theta(1)$.

In the following proposition, we prove that, under some conditions, $L^{\theta}_{\mathrm{d}}(w) = L^{\theta}_{\mathrm{cd}}(w)$ where θ is an antimorphic involution on X^*.

Lemma 3.1 ([8]) *Let θ be an antimorphic involution on X^* and $w \in X^+$. Then $L^{\theta}_{\mathrm{d}}(w) \subseteq L^{\theta}_{\mathrm{cd}}(w)$ and $K_\theta(1) \subseteq D_\theta(1)$.*

Proposition 3.1 *Let θ be an antimorphic involution on X^* and $w \in X^+$. Then $L^{\theta}_{\mathrm{d}}(w) = L^{\theta}_{\mathrm{cd}}(w)$ for every $w \in D_\theta(1) \cap K_\theta(1)$.*

Proof. Let θ be an antimorphic involution on X^* and $w \in X^+$. Let $w \in D_\theta(1) \cap K_\theta(1)$. By Lemma 3.1, $K_\theta(1) \subseteq D_\theta(1)$. It follows that $w \in K_\theta(1)$. Suppose that $L^{\theta}_{\mathrm{d}}(w) \subset L^{\theta}_{\mathrm{cd}}(w)$. By the definition of involution pseudoknot-unbordered words, we have $L^{\theta}_{\mathrm{cd}}(w) = \{\lambda\}$, that is, $|L^{\theta}_{\mathrm{cd}}(w)| = 1$. This implies that $|L^{\theta}_{\mathrm{d}}(w)| < 1$, a contradiction. Then by Lemma 3.1 again, we have $L^{\theta}_{\mathrm{d}}(w) = L^{\theta}_{\mathrm{cd}}(w)$. #

The set of all cyclic permutations of w is $\mathrm{Cp}(w) = \{yx | w = xy, x, y \in X^*\}$. For instance, let $w = abba$. Then $\mathrm{Cp}(w) = \{abba, aabb, baab, bbaa\}$.

Proposition 3.2 *Let $X = \{a, b\}$, θ be an antimorphic involution on X^* with $\theta(a) = b, \theta(b) = a$, and $w \in X^+$. Then $L^{\theta}_{\mathrm{d}}(w) = L^{\theta}_{\mathrm{cd}}(w)$ if for every $x, y \in X^*$ such that $xy \leq_{\mathrm{p}} w$ one has $\theta(yx) \not\leq_{\mathrm{s}} w$.*

Proof. Let $w \in X^+$ and θ be an antimorphic involution on X^* with $\theta(a) = b, \theta(b) = a$. Let $u = xy$ and $v = yx \in \mathrm{Cp}(u)$ for some $x, y \in X^*$. If $xy \leq_{\mathrm{p}} w$ such that $\theta(yx) \not\leq_{\mathrm{s}} w$, then from the definition of involution pseudoknot-bordered words, $L^{\theta}_{\mathrm{cd}}(w) = \{\lambda\}$. By Lemma 3.1, $L^{\theta}_{\mathrm{d}}(w) \subseteq L^{\theta}_{\mathrm{cd}}(w)$ implies that $L^{\theta}_{\mathrm{d}}(w) = L^{\theta}_{\mathrm{cd}}(w)$. #

For a word $w \in X^+$, let $\mathrm{Pref}(w) = \{v \in X^+ \mid v \leq_{\mathrm{p}} w\}$ and $\mathrm{Suff}(w) = \{v \in X^+ \mid v \leq_{\mathrm{s}} w\}$.

Proposition 3.3 *Let θ be an antimorphic involution on X^* and $w \in X^+$. If $L^{\theta}_{\mathrm{d}}(w) = L^{\theta}_{\mathrm{cd}}(w)$, then one of the following conditions holds:*

(1) $w \in K_\theta(1)$.

(2) $w \notin L^\theta_{\mathrm{cd}}(w)$ *and* $\theta(\mathrm{Cp}(\mathrm{Pref}(w))) \cap \mathrm{Suff}(w) = \theta(\mathrm{Pref}(w)) \cap \mathrm{Suff}(w) \neq \emptyset$.

Proof. Let θ be an antimorphic involution on X^* and $w \in X^+$. Assume that $L^\theta_{\mathrm{d}}(w) = L^\theta_{\mathrm{cd}}(w)$. If $L^\theta_{\mathrm{d}}(w) = L^\theta_{\mathrm{cd}}(w) = \{\lambda\}$, then $w \in D_\theta(1)$ and $w \in K_\theta(1)$. Thus Condition (1) holds. If $L^\theta_{\mathrm{d}}(w) = L^\theta_{\mathrm{cd}}(w) \neq \{\lambda\}$, then there exists a word $u \in X^+$ such that all the following conditions are true.

(a) $w = u\alpha_1 = \beta_1\theta(u)$ for some $\alpha_1, \beta_1 \in X^+$.

(b) $w = xy\alpha_2 = \beta_2\theta(yx)$ for some $x, y \in X^*$ with $u = xy \neq \lambda$ and $\alpha_2, \beta_2 \in X^*$.

From condition (a), since $\alpha_1, \beta_1 \in X^+$, we have $u \neq w$. It follows that $w \notin L^\theta_{\mathrm{d}}(w)$. This in conjunction with $L^\theta_{\mathrm{d}}(w) = L^\theta_{\mathrm{cd}}(w)$ yields that $w \notin L^\theta_{\mathrm{cd}}(w)$. Moreover form condition (b), there exists a word $u = xy \in X^+$ such that $\theta(u) \in \theta(\mathrm{Cp}(\mathrm{Pref}(w))) \cap \mathrm{Suff}(w)$. Since $L^\theta_{\mathrm{d}}(w) = L^\theta_{\mathrm{cd}}(w)$, it implies that $\theta(u) \in \theta(\mathrm{Pref}(w)) \cap \mathrm{Suff}(w)$. Thus $\theta(\mathrm{Cp}(\mathrm{Pref}(w))) \cap \mathrm{Suff}(w) = \theta(\mathrm{Pref}(w)) \cap \mathrm{Suff}(w) \neq \emptyset$. Then Condition (2) holds. #

A word u is called palindrome if it is the mirror image of itself. The notion of involution palindrome (or θ-palindrome) was studied in ([6], [9]). A language consisting of involution palindrome words is defined as an involution palindrome language. Besides involution palindrome words being considered, some algebraic properties of skew involution palindrome words are studied in [2]. We investigate the relation between θ-palindrome words and θ-pk-bordered words in the following context.

Lemma 3.2 ([8]) *Let θ be an antimorphic involution on X^*, and x, y be θ-palindromes such that $xy \neq \lambda$. If a word $u \in X^+$ has xy as both its prefix and suffix, then u is θ-pk-bordered.*

Lemma 3.3 ([2]) *Let θ be an antimorphic involution on X^*. A word w is skew θ-palindrome if and only if w is a product of two θ-palindrome words.*

From the above lemmata, the following result is clear.

Lemma 3.4 *Let θ be an antimorphic involution on X^*. Then every nonempty skew θ-palindrome word is θ-pk-bordered.*

Proof. Let $w \in X^+$ be a skew θ-palindrome word. By Lemma 3.3, there exist x, y are θ-palindrome words such that $w = xy$. That is, xy is a prefix of w and also a suffix of w. This in conjunction with Lemma 3.2 yields that w is θ-pk-bordered. #

From Lemma 3.4 and Proposition 3.3, we now have the following corollary.

Corollary 3.1 *Let θ be an antimorphic involution on X^* and $w \in X^+$. If w is a skew θ-palindrome word, then $L^\theta_{\mathrm{d}}(w) \neq L^\theta_{\mathrm{cd}}(w)$.*

Since a θ-palindrome word is a skew θ-palindrome word, the following result is also true.

Corollary 3.2 *Let θ be an antimorphic involution on X^* and $w \in X^+$. If w is a θ-palindrome word, then $L^\theta_{\mathrm{d}}(w) \neq L^\theta_{\mathrm{cd}}(w)$.*

Let $w \notin Q$. Then $w = f^k, f \in Q, k \geq 2$. The primitive root of w is denoted by $\sqrt{w} = f$.

Lemma 3.5 ([7]) *Let θ be a morphic or an antimorphic involution on X^*. For all $w \in X^+$, w is θ-palindrome if and only if $\sqrt{w}$ is θ-palindrome.*

Note that θ-palindrome words are θ-pk-bordered. By Lemma 3.5, the primitive root of words being θ-palindrome implies that those words are θ-pk-bordered. When we consider the case of θ-pk-unbordered words, the the primitive root of words being θ-pk-unbordered does not imply that those words are θ-pk-unbordered. For instance, let θ be an antimorphic involution on X^* with $\theta(a) = b, \theta(b) = a$ and let $w = aabbbbaba$. Then $w \in K_\theta(1)$ and $w^2 \notin K_\theta(1)$. That is, w is a θ-pk-unbordered word and w^2 is a θ-pk-bordered word. It implies that $\sqrt{w}$ may be a θ-pk-unbordered word for a θ-pk-bordered word w. In the following proposition, we will show that for every θ-pk-unbordered word w, $\sqrt{w}$ is also a θ-pk-unbordered. For instance, a nonprimitive θ-pk-unbordered word $w = a^2$, where $a \in X$. It is clear that $w, \sqrt{w}$ are θ-pk-unbordered words.

Proposition 3.4 *Let θ be an antimorphic involution on X^* and $w \in X^+$. Then $\sqrt{w} \in K_\theta(1)$ for every $w \in K_\theta(1)$.*

Proof. Let θ be an antimorphic involution on X^* and $w \in X^+$. If $w \in Q \cap K_\theta(1)$, then $w = \sqrt{w}$. It follows that $\sqrt{w} \in K_\theta(1)$. Now we consider $w \in K_\theta(1) \not\subseteq Q$. Let $w = f^k$ for some $f \in Q$ and $k \geq 2$. Then $\sqrt{w} = f$. Since $w \in K_\theta(1)$, w has no θ-pk-border with length $1 \leq m \leq \lg(w) = k\lg(f)$. It follows that w has no θ-pk-border with length $m \leq \lg(\sqrt{w}) = \lg(f)$. This implies that $\sqrt{w}$ has no θ-pk-border with length $1 \leq m' \leq \lg(f)$. Thus $\sqrt{w} \in K_\theta(1)$. #

Proposition 3.5 *Let θ be a morphic involution on X^*. For all $w \in X^+$, $\theta(L^\theta_{\mathrm{cd}}(w)) = L^\theta_{\mathrm{cd}}(\theta(w))$.*

Proof. Let θ be a morphic involution on X^* and $w \in X^+$. Let $u \in L^\theta_{\mathrm{cd}}(w)$. Then $\theta(u) \in \theta(L^\theta_{\mathrm{cd}}(w))$. By the definition of involution pseudoknot-border, we have $w = xy\alpha = \beta\theta(yx)$ where $u = xy \neq \lambda$ for some $x, y, \alpha, \beta \in X^*$. When θ is a morphic involution on X^*, we have

$$\theta(w) = \theta(xy\alpha) = \theta(\beta\theta(yx))$$

$$\Leftrightarrow \theta(w) = \theta(x)\theta(y)\theta(\alpha) = \theta(\beta)\theta(\theta(yx)) = \theta(\beta)\theta(\theta(y)\theta(x)).$$

Since $\theta(y)\theta(x) \in \mathrm{Cp}(\theta(x)\theta(y))$, it follows that $\theta(x)\theta(y) = \theta(xy) = \theta(u) \in L^\theta_{\mathrm{cd}}(\theta(w))$. Thus $\theta(L^\theta_{\mathrm{cd}}(w)) \subseteq L^\theta_{\mathrm{cd}}(\theta(w))$. Moreover, let $u' \in L^\theta_{\mathrm{cd}}(\theta(w))$. Then $\theta(w) = x'y'\alpha' = \beta'\theta(y'x')$ where $u' = x'y' \neq \lambda$ for some $x', y', \alpha', \beta' \in X^*$. When θ is a morphic involution on X^*, we have

$$\theta(\theta(w)) = w = \theta(x'y'\alpha') = \theta(\beta'\theta(y'x'))$$

$$\Leftrightarrow w = \theta(x')\theta(y')\theta(\alpha') = \theta(\beta')\theta(\theta(y'x')) = \theta(\beta')\theta(\theta(y')\theta(x')).$$

Since $\theta(y')\theta(x') \in \mathrm{Cp}(\theta(x')\theta(y'))$, we have $\theta(x')\theta(y') = \theta(x'y') = \theta(u') \in L^\theta_{\mathrm{cd}}(w)$. It follows that $\theta(\theta(u')) = u' \in \theta(L^\theta_{\mathrm{cd}}(w))$. Thus $L^\theta_{\mathrm{cd}}(\theta(w)) \subseteq \theta(L^\theta_{\mathrm{cd}}(w))$. Therefore, we have $\theta(L^\theta_{\mathrm{cd}}(w)) = L^\theta_{\mathrm{cd}}(\theta(w))$. #

Proposition 3.5 is not true when θ is an antimorphic involution on X^*. For instance, let θ be an antimorphic involution on X^* where $X = \{a, b\}$ with $\theta(a) = b$ and $\theta(b) = a$. Let $w = aabbba$. There is a word $u = aab$ where $x = aa$ and $y = b$. Then $u \in L^\theta_{\mathrm{cd}}(w)$. It follows that $\theta(u) = abb \in \theta(L^\theta_{\mathrm{cd}}(w))$. When θ is an antimorphic involution on X^*, we have $\theta(w) = baaabb$. This follows that $baa \in L^\theta_{\mathrm{cd}}(\theta(w))$. Thus $\theta(L^\theta_{\mathrm{cd}}(w)) \neq L^\theta_{\mathrm{cd}}(\theta(w))$.

Proposition 3.6 *Let θ be an antimorphic involution on X^*. For all $w \in X^+$, $L^\theta_{\mathrm{cd}}(w) \subseteq \mathrm{Cp}(L^\theta_{\mathrm{cd}}(\theta(w)))$.*

Proof. Let θ be an antimorphic involution on X^* and $w \in X^+$. Let $u \in L^{\theta}_{\text{cd}}(w)$. By the definition of involution pseudoknot-border, we have $w = xy\alpha = \beta\theta(yx)$ where $u = xy \neq \lambda$ for some $x, y, \alpha, \beta \in X^*$. When θ is an antimorphic involution on X^*, we have

$$\theta(w) = \theta(xy\alpha) = \theta(\beta\theta(yx))$$

$$\Leftrightarrow \theta(w) = \theta(\alpha)\theta(xy) = \theta(\theta(yx))\theta(\beta) = yx\theta(\beta).$$

Then $yx \in L^{\theta}_{\text{cd}}(\theta(w))$. This follows that $u = xy \in \text{Cp}(L^{\theta}_{\text{cd}}(\theta(w)))$. Thus $L^{\theta}_{\text{cd}}(w) \subseteq \text{Cp}(L^{\theta}_{\text{cd}}(\theta(w)))$. #

Note that if a word w is singleton, then it is θ-pk-unbordered when θ is a morphic or an antimorphic involution on X^*. By the definition of θ-pk-bordered word, we get the characteristic of nonempty θ-pk-bordered words in the following lemma.

Lemma 3.6 *Let θ be a morphic or an antimorphic involution on X^* and $w \in X^+$ with $\lg(w) \geq 2$. Then w is θ-pk-bordered if and only if $\theta(\text{Cp}(\text{Pref}(w))) \cap \text{Suff}(w) \neq \emptyset$.*

Lemma 3.7 *Let θ be an antimorphic involution on X^* and $w \in X^+$ with $\lg(w) \geq 2$. Then w is θ-pk-bordered if and only if $\theta(w)$ is θ-pk-bordered.*

Proof. Let θ be an antimorphic involution on X^* and $w \in X^+$ with $\lg(w) \geq 2$. Let w is θ-pk-bordered. Then $w = u\alpha = \beta\theta(v)$ for some $u, v, \alpha, \beta \in X^*$ where $v \in \text{Cp}(u)$. Let $u = xy$ for some $x, y \in X^*$ with $xy \neq \lambda$. Then we have $w = xy\alpha = \beta\theta(yx)$. This follows that $\theta(w) = \theta(xy\alpha) = \theta(\beta\theta(yx)) = \theta^2(yx)\theta(\beta)$. Thus $\theta(w) = yx\theta(\beta) = \theta(\alpha)\theta(xy)$. By the definition of θ-pk-bordered word, $\theta(w)$ is θ-pk-bordered. Moreover, the converse is true. #

In the following proposition, we provide a procedure to create θ-pk-bordered words.

Lemma 3.8 ([8]) *Let θ be an antimorphic involution on X^*. If a word $w \in X^+$ has a θ-pk-border of length n, then, for every $a \in X$, the number of occurrences of the letter a in the prefix of length n of w is equal to the number of occurrences of the letter $\theta(a)$ in the suffix of length n of w.*

Proposition 3.7 *Let θ be an antimorphic involution on X^* and for all $a \in X$ let $a \neq \theta(a)$. Let $w \in X^+$ with $\lg(w) \geq 2$. Then w is θ-pk-bordered if and only if one of the following statements is true:*

(1) $w = az\theta(a)$ for some $a \in X$ and $z \in X^*$.

(2) $w = xy$ for some $x, y \in X^+$ with $x = \theta(x)$ and $y = \theta(y)$.

(3) $w = xyz\theta(x)\theta(y)$ for some $x, y \in X^+$ and $z \in X^*$.

(4) $w = x_1x_2\theta(x_1)yx_1$ for some $x_1, x_2, y \in X^+$ with $x_2 = \theta(x_2)$ and $y = \theta(y)$.

(5) $w = x_1x_2\theta(x_1)x_1$ for some $x_1, x_2 \in X^+$ with $x_2 = \theta(x_2)$.

(6) $w = x_{11}x_{12}x_2\theta(x_{12})\theta(x_{11})x_{12}$ for some $x_{11}, x_{12}, x_2 \in X^+$ with $x_2 = \theta(x_2)$.

(7) $w = x_1x_2\theta(x_2)\theta(x_1)x_2$ for some $x_1, x_2 \in X^+$.

(8) $w = x\theta(x)x$ for some $x \in X^+$.

(9) $w = \theta(x_1)x_1x_2\theta(x_1)$ for some $x_1, x_2 \in X^+$ with $x_2 = \theta(x_2)$.

(10) $w = \theta(x_{21})x_1x_{21}\theta(x_{22})\theta(x_{21})\theta(x_1)$ for some $x_1, x_{21}, x_{22} \in X^+$ with $x_{22} = \theta(x_{22})$.

(11) $w = \theta(x_2)x_1x_2\theta(x_2)\theta(x_1)$ for some $x_1, x_2 \in X^+$.

(12) $w = x_1\theta(y_2)y_1y_2\theta(x_1)\theta(y_2)\theta(y_1)$ for some $x_1, y_1, y_2 \in X^+$.

Proof. Let θ be an antimorphic involution on X^* and for all $a \in X$ let $a \neq \theta(a)$. Let $w \in X^+$ with $\lg(w) \geq 2$. Since for all $a \in X$, $a \neq \theta(a)$, there exists $a \in X$ such that $\theta(a) = b$, where $a \neq b \in X$.

($\Rightarrow$) Let w is θ-pk-bordered. Then $w = u\alpha = \beta\theta(v)$ for some $u, v, \alpha, \beta \in X^*$ where $v \in \mathrm{Cp}(u)$. Let $u = xy$ for some $x, y \in X^*$ with $xy \neq \lambda$. Then we have $w = xy\alpha = \beta\theta(yx)$. We consider the following cases:

(1) $x = \lambda$ or $y = \lambda$. We consider $x = \lambda$. Then $w = y\alpha = \beta\theta(y)$. Let $y = ar$ for some $a \in X, r \in X^*$. Then $\theta(y) = \theta(ar) = \theta(r)\theta(a)$. Thus $w = ar\alpha = \beta\theta(r)\theta(a)$. Since $\lg(w) \geq 2$, this follows that $w = az\theta(a)$ for some $a \in X$ and $z \in X^*$. Hence Condition (1) holds. When $y = \lambda$, the proof is same as $x = \lambda$.

(2) $x, y \neq \lambda$. If $\alpha = \beta = \lambda$, then $w = xy = \theta(yx) = \theta(x)\theta(y)$. This implies that $x = \theta(x)$ and $y = \theta(y)$. Hence Condition (2) holds. By Lemma 3.8, cases $\alpha \neq \lambda, \beta = \lambda$ or $\alpha = \lambda, \beta \neq \lambda$ are omitted. We only consider the case: $\alpha \neq \lambda$ and $\beta \neq \lambda$. Then there are the following subcases: $\lg(u) \leq \frac{1}{2}\lg(w)$ or $\lg(u) > \frac{1}{2}\lg(w)$. When $\lg(u) \leq \frac{1}{2}\lg(w)$, we have $\lg(u) = \lg(xy) = \lg(\theta(x)\theta(y)) \leq \frac{1}{2}\lg(w)$. Since $w = xy\alpha = \beta\theta(yx) = \beta\theta(x)\theta(y)$, this in conjunction with $\lg(\alpha) = \lg(\beta) \geq \frac{1}{2}\lg(w)$ yields that $w = xyz\theta(x)\theta(y)$ for some $z \in X^*$. Hence Condition (3) holds. When $\lg(u) > \frac{1}{2}\lg(w)$, from the statement $xy\alpha = \beta\theta(x)\theta(y)$, there are the following subcases:

(2-1) $\lg(\beta) < \lg(x)$. There exist $x_1, x_2 \in X^+$ such that $x = x_1x_2$ and $\beta = x_1$. Then $\theta(x)\theta(y) = x_2y\alpha$. Since $\theta(x) = \theta(x_2)\theta(x_1)$, this implies that $x_2 = \theta(x_2)$ and $\theta(x_1)\theta(y) = y\alpha$. Next, we consider the statement $\theta(x_1)\theta(y) = y\alpha$. If $\lg(\theta(x_1)) < \lg(y)$, then there exist $y_1, y_2 \in X^+$ such that $y = y_1y_2$, $\theta(x_1) = y_1$ and $\theta(y) = y_2\alpha$. Since $\theta(y) = \theta(y_2)\theta(y_1)$, this follows that $y_2 = \theta(y_2)$ and $\theta(y_1) = \alpha$. Thus $w = xy\alpha = x_1x_2y_1y_2\theta(y_1) = x_1x_2\theta(x_1)y_2x_1$ with $x_2 = \theta(x_2)$ and $y_2 = \theta(y_2)$. Hence Condition (4) holds. If $\lg(\theta(x_1)) = \lg(y)$, then $\theta(x_1) = y$ and $\theta(y) = \alpha$. Thus $w = xy\alpha = x_1x_2y\theta(y) = x_1x_2\theta(x_1)x_1$ with $x_2 = \theta(x_2)$. Hence Condition (5) holds. If $\lg(\theta(x_1)) > \lg(y)$, then there exist $x_{11}, x_{12} \in X^+$ such that $x_1 = x_{11}x_{12}$, $\theta(x_{12}) = y$, and $\theta(x_{11})\theta(y) = \alpha$. Thus $w = xy\alpha = x_1x_2y\theta(x_{11})\theta(y) = x_{11}x_{12}x_2\theta(x_{12})\theta(x_{11})x_{12}$ with $x_2 = \theta(x_2)$. Hence Condition (6) holds.

(2-2) $\lg(\beta) = \lg(x)$. We have $\beta = x$ and $y\alpha = \theta(x)\theta(y)$. Next, we consider the statement $y\alpha = \theta(x)\theta(y)$. If $\lg(y) < \lg(\theta(x))$, then there exist $x_1, x_2 \in X^+$ such that $x = x_1x_2$, $\theta(x_2) = y$, and $\theta(x_1)\theta(y) = \alpha$. Thus $w = xy\alpha = x\theta(x_2)\theta(x_1)\theta(y) = x_1x_2\theta(x_2)\theta(x_1)x_2$. Hence Condition (7) holds. If $\lg(y) = \lg(\theta(x))$, then $\theta(x) = y$ and $\theta(y) = \alpha$. Thus $w = xy\alpha = xy\theta(y) = x\theta(x)x$. Hence Condition (8) holds. If $\lg(y) > \lg(\theta(x))$, then there exist $y_1, y_2 \in X^+$ such that $y = y_1y_2$, $\theta(x) = y_1$ and $\theta(y) = y_2\alpha$. Since $y = y_1y_2$, we have $\theta(y) = \theta(y_2)\theta(y_1) = y_2\alpha$. This follows that $\theta(y_2) = y_2$ and $\theta(y_1) = \alpha$. Thus $w = xy\alpha = \theta(y_1)y_1y_2\theta(y_1)$ with $\theta(y_2) = y_2$. Hence Condition (9) holds.

(2-3) $\lg(\beta) > \lg(x)$. There exist $y_1, y_2 \in X^+$ such that $y = y_1y_2$, $\beta = xy_1$, and $\theta(x)\theta(y) = y_2\alpha$. Next, we consider the statement $\theta(x)\theta(y) = y_2\alpha$. If $\lg(\theta(x)) < \lg(y_2)$, then there exist $y_{21}, y_{22} \in X^+$ such that $y_2 = y_{21}y_{22}$, $\theta(x) = y_{21}$, and $\theta(y) = y_{22}\alpha$. Since $\theta(y) = \theta(y_{22})\theta(y_{21})\theta(y_1)$, this in conjunction with $\theta(y) = y_{22}\alpha$ yields that $\theta(y_{21})\theta(y_1) = \alpha$ and $\theta(y_{22}) = y_{22}$. Thus $w = xy\alpha = xy_1y_{21}y_{22}\alpha = xy_1y_{21}\theta(y) = \theta(y_{21})y_1y_{21}\theta(y_{22})\theta(y_{21})\theta(y_1)$ with $y_{22} = \theta(y_{22})$. Hence Condition (10) holds. If $\lg(\theta(x)) = \lg(y_2)$, then $\theta(x) = y_2$ and $\theta(y) = \alpha$. Thus $w = xy\alpha = xy\theta(y) = \theta(y_2)y_1y_2\theta(y_2)\theta(y_1)$. Hence Condition (11) holds. If $\lg(\theta(x)) > \lg(y_2)$, then there exist $x_1, x_2 \in X^+$ such that $x = x_1x_2$, $\theta(x_2) = y_2$, and $\theta(x_1)\theta(y) = \alpha$. Thus $w = xy\alpha = xy_1y_2\theta(x_1)\theta(y) = x_1\theta(y_2)y_1y_2\theta(x_1)\theta(y_2)\theta(y_1)$. Hence Condition (12) holds.

($\Leftarrow$) The converse is obvious. #

From Proposition 3.7, it is simple to create θ-pk-bordered words. For instance, if condition (5) of Proposition 3.7 is used, then we can let $x_1 = a, x_2 = aabb$ when θ is an antimorphic involution on X^* with $\theta(a) = b, \theta(b) = a$. Thus $w = aaabbba$ is a θ-pk-bordered word where θ-pk-border is $aaab$. Let R_θ be the set of all θ-palindrome words over X.

Lemma 3.9 ([7]) *Let θ be an antimorphic involution on X^* and for all $a \in X$ let $a \neq \theta(a)$. Then for $w \in X^+$, $w \in R_\theta$ if and only if $w = xy\theta(x), x \in X^+, y \in X^*$ with $y \in R_\theta$.*

Proposition 3.8 *Let θ be an antimorphic involution on X^* and for all $a \in X$ let $a \neq \theta(a)$. If $w \in X^+$ is a θ-palindrome word, then w is θ-pk-bordered.*

Proof. Let θ be an antimorphic involution on X^* and for all $a \in X$ let $a \neq \theta(a)$. Let $w \in X^+$ be a θ-palindrome word. By Lemma 3.9, there exists $x \in X^+, y \in X^*$ such that $w = xy\theta(x)$ with $y \in R_\theta$. By the definition of θ-pk-border, there exists a θ-pk-border $x \in X^+$ such that w is θ-pk-bordered. #

4 The Density of θ-bordered and θ-pk-bordered Languages

Let θ be an antimorphic involution with not identity on X^*. Recall that $D_\theta(1)$ contains all θ-unbordered words and $\bigcup_{i \geq 2} D_\theta(i)$ contains all θ-bordered words. Any nonempty subset of $\bigcup_{i \geq 2} D_\theta(i)$ is called a θ-bordered language and any nonempty subset of $D_\theta(1)$ is called a θ-unbordered language. Similarly, recall that $K_\theta(1)$ contains all θ-pk-unbordered words and $\bigcup_{i \geq 2} K_\theta(i)$ contains all θ-pk-bordered words. Any nonempty subset of $\bigcup_{i \geq 2} K_\theta(i)$ is called a θ-pk-bordered language and any nonempty subset of $K_\theta(1)$ is called a θ-pk-unbordered language. A language $L \subseteq X^*$ is *dense* if for any $w \in X^*$, there exist $x, y \in X^*$ such that $xwy \in L$ ([1]). That is, for every $w \in X^*$, $X^*wX^* \cap L \neq \emptyset$. In this section we will investigate the density of $D_\theta(i),\ K_\theta(i),\ i \geq 1$.

Lemma 4.1 *Let θ be an antimorphic involution on X^* and for all $a \in X$ let $a \neq \theta(a)$. Let w be a θ-bordered word. Then there exists a θ-border v of w with $\lg(v) \leq \frac{1}{2}\lg(w)$.*

Proof. Let θ be an antimorphic involution on X^* and for all $a \in X$ let $a \neq \theta(a)$. Let w be a θ-bordered word. Then there exists $u \in X^+$ such that $w = u\alpha = \beta\theta(u)$ for some $\alpha, \beta \in X^+$. If $\lg(u) \leq \frac{1}{2}\lg(w)$, then we are done. Now assume that $\lg(u) > \frac{1}{2}\lg(w)$. We have $\lg(\beta) < \frac{1}{2}\lg(w)$. There exists $x \in X^+$ such that $u = \beta x$ and $\theta(u) = x\alpha$. This in conjunction with θ being an antimorphic involution yields that $x\alpha = \theta(u) = \theta(\beta x) = \theta(x)\theta(\beta)$. It follows that $x = \theta(x)$ and $\alpha = \theta(\beta)$. Then $w = u\alpha = \beta\theta(u) = u\theta(\beta)$. Take $v = \beta$. Thus v is a θ-border of w and $\lg(v) < \frac{1}{2}\lg(w)$. We are done. #

Proposition 4.1 *Let θ be an antimorphic involution on X^* and for all $a \in X$ let $a \neq \theta(a)$. Then $D_\theta(i)$ for every $i \geq 1$ are dense.*

Proof. Let θ be an antimorphic involution on X^* and for all $a \in X$ let $a \neq \theta(a)$. There exists $a \in X$ such that $\theta(a) = b$, where $a \neq b \in X$. We will show that $D_\theta(i), i \geq 1$ are dense. First, we show that $D_\theta(1)$ is dense. By the definition of antimorphic involution on X^*, we have $awa \in D_\theta(1)$. Hence $D_\theta(1)$ is dense. Next, we show that $D_\theta(i),\ i \geq 2$ are dense. If $w = \lambda$, then $a^i wab^{i-1} = a^{i+1}b^{i-1}$. It is clear that $a^j \in L^\theta_{\mathrm{d}}(a^{i+1}b^{i-1})$ for $0 \leq j \leq i-1$. Then every word whose length is greater than $i-1$ is not a θ-border of $a^{i+1}b^{i-1}$. Thus $L^\theta_{\mathrm{d}}(a^{i+1}b^{i-1}) = \{\lambda, a, \ldots, a^{i-1}\}$ for some $i \geq 2$. Hence $X^+wX^+ \cap D_\theta(i) \neq \emptyset,\ i \geq 2$. Assume that $w \in X^+$. Let $n = lg(w)$ and let $x = a^{2n+i-1}, y = a^{2n}b^{i-1}$. Then it implies that $L^\theta_{\mathrm{d}}(xwy) = \{\lambda, a, \ldots, a^{i-1}\}$. Indeed, on the contrary, assume that $xwy = u\alpha = \beta\theta(u)$ for some $u \in X^+$ with $\lg(u) > i-1$. By Lemma 4.1, there are the following cases:

(1) $i-1 < \lg(u) \leq 2n+i-1$. In this case, $u = a^{k+i-1}$ with $0 < k \leq 2n$ and $\theta(u) = a^k b^{i-1}$. This follows that $a^k b^{i-1} = \theta(u) = a^{k+i-1}$, a contradiction.

(2) $2n+i-1 < \lg(u) \leq 2n+i-1+\frac{1}{2}n$. That is, $\lg(a^{2n+i-1}) = lg(x) < \lg(u) \leq \lg(x)+\frac{1}{2}\lg(w)$. In this case, there exists $p, q \in X^+$ with $p \leq_{\rm p} w$ and $q \leq_{\rm s} w$ such that $u = a^{2n+i-1}p$ and $\theta(u) = qa^{2n}b^{i-1}$. This follows that $qa^{2n}b^{i-1} = \theta(u) = \theta(a^{2n+i-1}p) = \theta(p)\theta(a^{2n+i-1}) = \theta(p)b^{2n+i-1}$, a contradiction.

By above discussion, u is not a θ-border of $a^{2n+i-1}wa^{2n}b^{i-1}$ with $lg(u) > i-1$. We have $L_{\rm d}^{\theta}(xwy) = \{\lambda, a, \ldots, a^{i-1}\}$. Hence $D_\theta(i),\ i \geq 1$ are dense. #

Note that for θ-pk-bordered words, Lemma 4.1 is not true. For instance, let $w = aba$. Then $w = ab \cdot a = a \cdot \theta(ba)$. That is, $ab \in L_{\rm cd}^{\theta}(w)$ and $\lg(ab) > \frac{1}{2}\lg(w)$, but $a \notin L_{\rm cd}^{\theta}(w)$. That is, nonempty θ-pk-border of $w = aba$ is greater than $\frac{1}{2}\lg(w)$. In the following proposition, we study the density of $K_\theta(i), i \geq 1$ for an antimorphic involution θ on X^* with $a \neq \theta(a)$ for all $a \in X$. First, we give the following lemma.

Lemma 4.2 [8] *Let θ be a antimorphic involution on X^*. Then $K_\theta(1)$, the set of all θ-pseudoknot-unbordered words over X^*, is a dense set.*

Proposition 4.2 *Let θ be an antimorphic involution on X^* and for all $a \in X$ let $a \neq \theta(a)$. Then $K_\theta(i)$ for $i \geq 1$ are dense.*

Proof. Let θ be an antimorphic involution on X^* and for all $a \in X$ let $a \neq \theta(a)$. There exists $a \in X$ such that $\theta(a) = b$, where $a \neq b \in X$. By Lemma 4.2, we have that $K_\theta(1)$ is a dense set. Thus we will show that $K_\theta(i)$ for $i \geq 2$ are dense. Let $\lg(w) = n$ and let $x = (ab)^{i-1}a^{2n+1}$, $y = a^{2n+1}(ba)^{i-1}$. It follows that $xwy \in K_\theta(i)$ for $i \geq 2$. Indeed, we will show that $L_{\rm cd}^{\theta}(xwy) = \{\lambda, ab, (ab)^2, \ldots, (ab)^{i-1}\}$. First, by the definition of involution pseudoknot-border, we have $(ab)^j \in L_{\rm cd}^{\theta}(xwy)$ and $(ab)^j a \notin L_{\rm cd}^{\theta}(xwy)$, for all $0 \leq j \leq i-1$. Next, we consider that for every $u \leq_{\rm p} xwy$ with $\lg(u) > i$. There are the following cases:

(1) $i < \lg(u) \leq 2n+1+(i-1)$. We have $u = (ab)^{i-1}a^k$ where $2 \leq k \leq 2n+1$. Let $u = x'y'$ where $x' = (ab)^{i-1}$ and $y' = a^k$. Then $\theta(x') = (ab)^{i-1}$ and $\theta(y') = b^k$. This implies that $xwy = (ab)^{i-1}a^{2n+1}wa^{2n+1}(ba)^{i-1} = (ab)^{i-1}a^k\alpha'' = \beta''(ab)^{i-1}b^k$ for some $\alpha'', \beta'' \in X^+$. We have $a = b$, a contradiction.

(2) $2n+1+(i-1) < \lg(u) \leq 2n+1+(i-1)+\lg(w)$. There exists $f \in X^+$ with $f \leq_{\rm p} w$ such that $u = (ab)^{i-1}a^{2n+1}f$. Let $u = x'y'$ where $x' = (ab)^{i-1}$ and $y' = a^{2n+1}f$. Then $\theta(x') = (ab)^{i-1}$ and $\theta(y') = \theta(f)b^{2n+1}$. This implies that $xwy = (ab)^{i-1}a^{2n+1}wa^{2n+1}(ba)^{i-1} = (ab)^{i-1}a^{2n+1}f\alpha'' = \beta''(ab)^{i-1}\theta(f)b^{2n+1}$ for some $\alpha'', \beta'' \in X^+$. We have $a = b$, a contradiction.

(3) $2n+1+(i-1)+\lg(w) < \lg(u) \leq 2(2n+1)+(i-1)+\lg(w)$. There exists an integer k with $1 \leq k \leq 2n+1$ such that $u = (ab)^{i-1}a^{2n+1}wa^k$. Let $u = x'y'$ where $x' = (ab)^{i-1}$ and $y' = a^{2n+1}wa^k$. Then $\theta(x') = (ab)^{i-1}$ and $\theta(y') = b^k\theta(w)b^{2n+1}$. This implies that $xwy = (ab)^{i-1}a^{2n+1}wa^{2n+1}(ba)^{i-1} = (ab)^{i-1}a^{2n+1}wa^k\alpha'' = \beta''(ab)^{i-1}b^k\theta(w)b^{2n+1}$ for some $\alpha'', \beta'' \in X^+$. We have $a = b$, a contradiction.

(4) $2(2n+1)+(i-1)+\lg(w) < \lg(u)$. There exists an integer j with $1 \leq j \leq i-1$ such that $u = (ab)^{i-1}a^{2n+1}wa^{2n+1}(ba)^j$. Let $u = x'y'$ where $x' = (ab)^{i-1}$ and $y' = a^{2n+1}wa^{2n+1}(ba)^j$. Then $\theta(x') = (ab)^{i-1}$ and $\theta(y') = (ba)^j b^{2n+1}\theta(w)b^{2n+1}$. This implies that $xwy = (ab)^{i-1}a^{2n+1}wa^{2n+1}(ba)^{i-1} = (ab)^{i-1}a^{2n+1}wa^k\alpha'' = \beta''(ba)^j b^{2n+1}\theta(w)b^{2n+1}$ for some $\alpha'', \beta'' \in X^+$. We have $a = b$, a contradiction.

By above discussion, u is not a θ-border of $(ab)^{i-1}a^{2n+1}wa^{2n+1}(ba)^{i-1}$. This implies that $L_{\rm cd}^{\theta}(xwy) = \{\lambda, ab, (ab)^2, \ldots, (ab)^{i-1}\}$. Hence $K_\theta(i), i \geq 1$ are dense. #

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